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LETTER TO THE EDITOR

# Analytic solution for a boundary value problem in the theory of Brownian motion

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**Abstract.** A closed-form formula for the solution of the first boundary value problem for the Kolmogorov equation is found by constructing explicitly its Green function.

The aim of this letter is to give an explicit solution to the first boundary value problem for the Kolmogorov equation. The Kolmogorov equation [1] is the simplest equation arising in the theory of Brownian motion and the problem is the following. Find the solution of the equation

$$Lf(x, u, t) \equiv \frac{\partial^2 f}{\partial u^2} - u\frac{\partial f}{\partial x} - \frac{\partial f}{\partial t} = q(x, u, t) \qquad x, t \in \mathbb{R}_+, u \in \mathbb{R} \tag{1}$$

with the initial condition

$$f(x, u, 0) = \varphi(x, u) \tag{2}$$

and the boundary condition (BC)

$$f(0, u, t) = 0 \qquad \text{for } u \geqslant 0. \tag{3}$$

We solve the problem (1)–(3) by finding explicitly its Green function. The BCs we shall consider for the Green function are more general having the following form

$$f(0, u, t) = kf(0, -u, t) \qquad \text{for } u > 0.$$

In physical applications $k$ is a non-negative parameter bounded by unity, $0 \leqslant k \leqslant 1$; $k = 0$ corresponds to absorption and $k = 1$ to reflection of particles upon the boundary $x = 0$. The extreme cases, $k = 0$ and $k = 1$ for the Fokker–Planck and Kramers equations, have been under study since the early 1940s, but until now the results have been rather meagre. Wang and Uhlenbeck in their review paper [2] quote the following as unsolved problems.

(i) The Brownian motion for particles in a gravitational field where one needs a reflecting boundary, say at $x = 0$, in order to prevent the particles from disappearing towards $x = \infty$.

(ii) The extension of the Smoluchowski method for finding the first passage time for two-dimensional Markov processes. For doing this it is necessary to introduce the idea of an absorbing boundary at $x = 0$.

The last time this type of problem arose was in the work of reaction kineticists [3–5] who tried to understand, for example, the dynamics of the aerocolloidal systems. The physical problem is to determine the coagulation coefficient which measures the probability of particles to come together and coalesce as a function of their size, viscosity, etc.

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Recently Marshall and Watson attempted to find the Green function with absorbing BC for the Fokker-Planck equation in a uniform force field [6, 7]. They succeeded only in finding its Laplace transform given as an infinite series. However, their result proved to be useful in solving some boundary layer problems in the theory of Brownian motion [7].

It will become clear from what follows that for solving such problems it is necessary to find Green functions that are multiform solutions of partial differential equations like (1). The method for obtaining multiform solutions of partial differential equations was invented by Sommerfeld at the end of the last century [8, 9] when he treated problems of diffraction and reflection of plane waves of light or sound incident on an opaque semi-infinite plane bounded by a straight edge. The method is essentially an extension of the image technique much used in problems of electrostatics and heat conduction [9].

Now our aim is to find the Green function of (1) for $x$, $t \in \mathbb{R}_+$, $u \in \mathbb{R}$, which gives the transition probability for a particle initially at $(y, v)$ and which satisfies the BC

$$G(x, u; y, v; 0+) = \delta(x-y)\delta(u-v) \tag{4}$$

$$G(0, u; y, v; t) = kG(0, -u; y, v; t) \qquad \text{for } u>0 \tag{5}$$

$$G(x, u; y, v; t) \to 0 \qquad \text{as } u \to +\infty \text{ and as } x \to +\infty. \tag{6}$$

Our starting point in finding the Green function of (1) with BC (4)-(6) is the fundamental solution of (1) which has the form

$$\phi(x, u; y, v; t) = \frac{\sqrt{3}}{2\pi t^2} \exp\{-[(X-X_0)^2 + (Y-Y_0)^2]/t\} \tag{7}$$

where

$$X = u - 3x/2t \qquad Y = \sqrt{3}\, x/2t$$

$$X_0 = -v/2 - 3y/2t \qquad Y_0 = \sqrt{3}/2(v + y/t).$$

In polar coordinates the relation (7) is written as

$$\phi(x, u; y, v; t) = \frac{\sqrt{3}}{2\pi t^2} \exp\left[-\frac{r^2 + r'^2 - 2rr' \cos(\vartheta - \vartheta')}{t}\right] \tag{7'}$$

where

$$X = r \cos \vartheta \qquad Y = r \sin \vartheta$$

$$X_0 = r' \cos \vartheta' \qquad Y_0 = r' \sin \vartheta'.$$

Following Carlslaw [9] we now construct a multiform Green function which has the form

$$g(\vartheta, \vartheta') = \frac{\sqrt{3}}{2\pi^{3/2} t^2} \exp\left[-\frac{r^2 + r'^2 - 2rr' \cos(\vartheta - \vartheta')}{t}\right] \int_{-\infty}^{a} e^{-\lambda^2}\, d\lambda \tag{8}$$

where

$$a = 2\sqrt{\frac{rr'}{t}} \cos\left(\frac{\theta - \theta'}{2}\right).$$

This function has the following properties:

(i) it is a solution of (1) when $q(x, u, t)=0$ and $t+|x-y|+|u-v|>0$;

(ii) on the two-sheeted Riemann surface obtained by cutting along the direction $\vartheta'+\pi$ from the origin to infinity it is uniform; in other words it is a periodic function in $\vartheta$, and of period $4\pi$;

(iii) on this surface it has only a singularity at the point $(r', \vartheta')$; on the first sheet, when $t=0$, $g(\vartheta, \vartheta')=\phi(x, u; y, v; t)$; on the other sheet $g=0$;

(iv) $g(\vartheta, \vartheta')\to 0$ as $r\to\infty$.

The idea used by Carlslaw in obtaining a formula like (8) is simple. Since $\phi(x, u; y, v; t)$ satisfies (1) when $q(x, u, t)=0$ and $t+|x-y|+|u-v|>0$ the function

$$g_0(x, y; u, v; t)=\frac{\sqrt{3}}{2\pi t^2}\int f(\alpha)\exp\left[-\frac{r^2+r'^2-2rr'\cos(\vartheta-\alpha)}{t}\right]\mathrm{d}\alpha$$

where $f(\alpha)$ is an arbitrary function, will also be a solution by the superposition principle, provided the above-stated conditions hold. For obtaining a multiform Green function $f(\alpha)$ is taken to be the Cauchy kernel in the form

$$f(\alpha)=\frac{1}{2\pi n}\frac{\mathrm{e}^{\mathrm{i}\alpha/n}}{\mathrm{e}^{\mathrm{i}\alpha/n}-\mathrm{e}^{\mathrm{i}\theta'/n}}$$

when we look for an $n$-sheeted Riemann surface, and the integral is taken over a path in the complex plane $\alpha$ composed of two curves extending to infinity and such that $\operatorname{Re}\cos(\theta-\alpha)>0$ there.

Now it is easy to verify that the functions

$$G_k(x, u; y, v; t)=g(\vartheta, \vartheta')-g(\vartheta, -\vartheta')+k[g(\vartheta+\pi, \vartheta')-g(\vartheta+\pi, -\vartheta')] \tag{9}$$

for $k\in[0, 1]$, satisfy the BC (4).

Thus the functions (9) give the Green functions for the Kolmogorov (1) with the BC (4)-(6).

As an application one finds easily the first passage time density as given by

$$p(y, v; t)=\int_{-\infty}^{0} uG_0(0+, u; y, v; t)\,\mathrm{d}u=\frac{\sqrt{3}}{2\pi^{3/2}t}\left(\frac{\sqrt{\pi r'}}{2}\sin\frac{\vartheta'}{2}\mathrm{e}^{-r'^2/2t}D_{-1/2}(r'\sqrt{2/t})\right.$$
$$\left.+\frac{r'\cos\vartheta'}{t}\int_0^{\infty}\mathrm{d}r\exp\left(-\frac{r^2+r'^2+2rr'\cos\vartheta'}{t}\right)\int_{-b}^{b}\mathrm{e}^{-\lambda^2}\,\mathrm{d}\lambda\right)$$

where $D_\nu(x)$ is the parabolic cylinder function and $b=2\sqrt{rr'/t}\sin\vartheta'/2$.

Now we can prove the following.

*Theorem.* The first boundary value problem (1)-(3) has a unique solution if the following hold true:

(1) the function $q(x, u, t)$ is continuous in all the variables $x$, $u$, $t$ and satisfies a Hölder condition in $x$ and $u$;

(2) the function $\varphi(x, u)$ is continuous and bounded everywhere;

(3) the explicit solution has the form

$$f(x, u, t)=-\int_0^t\mathrm{d}\tau\int_0^{\infty}\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}v\,G_0(x, u; y, v; t-\tau)q(y, v, \tau)$$
$$+\int_0^{\infty}\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}v\,G_0(x, u; y, v; t)\varphi(y, v).$$

A full proof of the above results, as well as applications of the method to other kinetic equations, will be given elsewhere.

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